MATH3280A Introductory Probability, 2014-2015
Solutions to HW5

P.246 Ex.8

Solution
The probability density function of X is given by

$$f_X(t) = \begin{cases} e^{-t} & \text{, if } t \geq 0 \\ 0 & \text{, if } t < 0. \end{cases}$$

Denote the distribution function of X by F_X.

$$Y = \begin{cases} X & \text{, if } X \leq 1 \\ \dfrac{1}{X} & \text{, if } X > 1. \end{cases}$$

Then the distribution function of Y is

$$\begin{aligned}
F_Y(t) &= P(Y \leq t) \\
&= P(Y \leq t | X \leq 1)P(X \leq 1) + P(Y \leq t | X > 1)P(X > 1) \\
&= P(X \leq t | X \leq 1)P(X \leq 1) + P(\frac{1}{X} \leq t | X > 1)P(X > 1) \\
&= \begin{cases} 0 & \text{, if } t \leq 0 \\ \frac{P(X \leq t \text{ and } X \leq 1)}{P(X \leq 1)}P(X \leq 1) + \frac{P(X \geq 1/t \text{ and } X > 1)}{P(X > 1)}P(X > 1)) & \text{, if } 0 < t < 1 \\ \frac{P(X \leq t \text{ and } X \leq 1)}{P(X \leq 1)}P(X \leq 1) + \frac{P(X \geq 1/t \text{ and } X > 1)}{P(X > 1)}P(X > 1)) & \text{, if } t \geq 1 \end{cases} \\
&= \begin{cases} 0 & \text{, if } t \leq 0 \\ P(X \leq t) + P(X \geq \frac{1}{t}) & \text{, if } 0 < t < 1 \\ P(X \leq 1) + P(X > 1) & \text{, if } t \geq 1 \end{cases} \\
&= \begin{cases} 0 & \text{, if } t \leq 0 \\ F_X(t) + 1 - F_X(\frac{1}{t}) & \text{, if } 0 < t < 1 \\ 1 & \text{, if } t \geq 1 \end{cases}
\end{aligned}$$

Then we have

$$\begin{aligned}
F'_Y(t) &= \begin{cases} 0 & \text{, if } t < 0 \text{ or } t > 1 \\ f_X(t) - (-\frac{1}{t^2})f_X(\frac{1}{t}) & \text{, if } 0 < t < 1 \end{cases} \\
&= \begin{cases} 0 & \text{, if } t < 0 \text{ or } t > 1 \\ e^{-t} + \dfrac{1}{t^2}e^{-\frac{1}{t}} & \text{, if } 0 < t < 1 \end{cases}
\end{aligned}$$

Define

$$f_Y(t) = \begin{cases} 0 & , \text{if } t < 0 \text{ or } t > 1 \\ e^{-t} + \frac{1}{t^2}e^{-\frac{1}{t}} & , \text{if } 0 \leq t \leq 1 \end{cases}$$

We can check that

$$\int_{-\infty}^{t} f_Y(y)dy = F_Y(t), \text{ for any } t \in \mathbb{R}.$$

Hence f_Y is a probability density function of Y.

□

P.267 Ex.10

Solution
The probability density function of θ is

$$f_\theta(t) = \begin{cases} \frac{1}{\pi} & , \text{if } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \\ 0 & , \text{if } t < -\frac{\pi}{2} \text{ or } t > \frac{\pi}{2}. \end{cases}$$

Denote the distribution function of θ by F_θ.

The distribution function of $X = \tan(\theta)$ is

$$\begin{aligned} F_X(t) &= P(\tan(\theta) \leq t) \\ &= P(\theta \leq \arctan(t)) \\ &= F_\theta(\arctan(t)) \text{ , for any } t \in \mathbb{R}. \end{aligned}$$

A probability density function of X is

$$\begin{aligned} f_X(t) = F_X'(t) = F_\theta'(\arctan(t)) \\ &= f_\theta(\arctan(t)) \frac{d}{dt} \arctan(t) \\ &= \frac{1}{\pi(1+t^2)} \text{ , for any } t \in \mathbb{R}. \end{aligned}$$

□

P.281 Ex.9

Solution
Let X be the random variable of the length of a steel sheet manufactured.
It is given that $X \sim N(75, 1^2)$.
We have $X - 75 \sim N(0, 1^2)$, the standard normal random variable.

The required probability is

$$\begin{aligned}
P(74.5 \leq X \leq 75.8) &= P(-0.5 \leq X - 75 \leq 0.8) \\
&= P(X - 75 \leq 0.8) - P(X - 75 < -0.5) \\
&= \Phi(0.8) - \Phi(-0.5) \\
&= \Phi(0.8) - (1 - \Phi(0.5)) \\
&\approx 0.7881 - (1 - 0.6915) \\
&= 0.4796.
\end{aligned}$$

□